arXiv:hep-ph/0006089v2 12 Jul 2000

Improved Conformal Mapping of the Borel Plane

U. D. Jentschura and G. Soff

Institut für Theoretische Physik, TU Dresden, 01062 Dresden, Germany

Email: jentschura@physik.tu-dresden.de, soff@physik.tu-dresden.de

<u>Abstract</u> The conformal mapping of the Borel plane can be utilized for the analytic continuation of the Borel transform to the entire positive real semi-axis and is thus helpful in the resummation of divergent perturbation series in quantum field theory. We observe that the convergence can be accelerated by the application of Padé approximants to the Borel transform expressed as a function of the conformal variable, i.e. by a combination of the analytic continuation via conformal mapping and a subsequent numerical approximation by rational approximants. The method is primarily useful in those cases where the leading (but not sub-leading) large-order asymptotics of the perturbative coefficients are known.

The problem of the resummation of quantum field theoretic series is of obvious importance in view of the divergent, asymptotic character of the perturbative expansions [1–3]. The convergence can be accelerated when additional information is available about large-order asymptotics of the perturbative coefficients [4]. In the example cases discussed in [4], the location of several poles in the Borel plane, known from the leading and next-to-leading large-order asymptotics of the perturbative coefficients, is utilized in order to construct specialized resummation prescriptions. Here, we consider a particular perturbation series, investigated in [5], where only the *leading* large-order asymptotics of the perturbative coefficients are known to sufficient accuracy, and the subleading asymptotics have – not yet – been determined. Therefore, the location of only a single pole – the one closest to the origin – in the Borel plane is available. In this case, as discussed in [6, 7], the (asymptotically optimal) conformal mapping of the Borel plane is an attractive method for the analytic continuation of the Borel transform beyond its circle of convergence and, to a certain extent, for accelerating the convergence of the Borel transforms. Here, we argue that the convergence of the transformation can be accelerated further when the Borel transforms, expressed as a function of the conformal variable which mediates the analytic continuation, are additionally convergence-accelerated by the application of Padé approximants.

First we discuss, in general terms, the construction of the improved conformal mapping of the Borel plane which is used for the resummation of the perturbation series defined in Eqs. (14) and (18) below. The method uses as input data the numerical values of a finite number of perturbative coefficients and the leading large-order asymptotics of the perturbative coefficients, which can, under appropriate circumstances, be derived from an empirical investigation of a finite number of coefficients, as it has been done in [5]. We start from an asymptotic, divergent perturbative expansion of a physical observable $f(g)$ in powers of a coupling parameter g,

$$f(g) \sim \sum_{n=0}^{\infty} c_n\, g^n\,, \tag{1}$$

and we consider the generalized Borel transform of the $(1, \lambda)$-type (see Eq. (4) in [4]),

$$f_{\mathrm{B}}^{(\lambda)}(u) \;\equiv\; f_{\mathrm{B}}^{(1,\lambda)}(u) \;=\; \sum_{n=0}^{\infty} \frac{c_n}{\Gamma(n+\lambda)}\, u^n\,. \tag{2}$$

The full physical solution can be reconstructed from the divergent series (1) by evaluating the Laplace-Borel integral, which is defined as

$$f(g) = \frac{1}{g^\lambda}\, \int_0^\infty \mathrm{d}u\, u^{\lambda-1}\, \exp\!\big(-u/g\big)\, f_{\mathrm{B}}^{(\lambda)}(u)\,. \tag{3}$$

The integration variable u is referred to as the Borel variable. The integration is carried out either along the real axis or infinitesimally above or below it (if Padé approximants are used for the analytic continuation, modified integration contours have been proposed [8]). The most prominent issue in the theory of the Borel resummation is the construction of an analytic continuation for the Borel transform (2) from a finite-order partial sum of the perturbation series (1), which we denote by

$$f_{\mathrm{B}}^{(\lambda),m}(u) = \sum_{n=0}^{m} \frac{c_n}{\Gamma(n+\lambda)}\, u^n\,. \tag{4}$$

The analytic continuation can be accomplished using the direct application of Padé approximants to the partial sums of the Borel transform $f_{\mathrm{B}}^{(\lambda),m}(u)$ [5, 8–10] or by a conformal mapping [6, 7, 11–13]. We now assume that the *leading* large-order asymptotics of the perturbative coefficients c_n defined in Eq. (1) is factorial, and that the coefficients display an alternating sign pattern. This indicates the existence of a singularity (branch point) along the negative real axis corresponding to the leading large-order growth of

2

the perturbative coefficients, which we assume to be at $u = -1$. For Borel transforms which have only a single cut in the complex plane which extends from $u = -1$ to $u = -\infty$, the following conformal mapping has been recommended as optimal [6],

$$z = z(u) = \frac{\sqrt{1+u} - 1}{\sqrt{1+u} + 1}.$$ (5)

Here, z is referred to as the conformal variable. The cut Borel plane is mapped unto the unit circle by the conformal mapping (5). We briefly mention that a large variety of similar conformal mappings have been discussed in the literature [14–19].

It is worth noting that conformal mappings which are adopted for doubly-cut Borel planes have been discussed in [6,7]. We do not claim here that it would be impossible to construct conformal mappings which reflect the position of more than two renormalon poles or branch points in the complex plane. However, we stress that such a conformal mapping is likely to have a more complicated mathematical structure than, for example, the mapping defined in Eq. (27) in [6]. Using the alternative methods described in [4], poles (branch points) in the Borel plane corresponding to the subleading asymptotics can be incorporated easily provided their position in the Borel plane is known. In a concrete example (see Table 1 in [4]), 14 poles in the Borel plane have been fixed in the denominator of the Padé approximant constructed according to Eqs. (53)–(55) in [4], and accelerated convergence of the transforms is observed. In contrast to the investigation [4], we assume here that only the *leading* large-order factorial asymptotics of the perturbative coefficients are known.

We continue with the discussion of the conformal mapping (5). It should be noted that for series whose leading singularity in the Borel plane is at $u = -u_0$ with $u_0 > 0$, an appropriate rescaling of the Borel variable $u \to |u_0| \, u$ is necessary on the right-hand side of Eq. (3). Then, $f_{\mathrm{B}}^{(\lambda)}(|u_0| \, u)$ as a function of u has its leading singularity at $u = -1$ (see also Eq. (41.57) in [2]). The Borel integration variable u can be expressed as a function of z as follows,

$$u(z) = \frac{4 z}{(z-1)^2}.$$ (6)

The mth partial sum of the Borel transform (4) can be rewritten, upon expansion of the u in powers of z, as

$$f_{\mathrm{B}}^{(\lambda),m}(u) = f_{\mathrm{B}}^{(\lambda),m}\big(u(z)\big) = \sum_{n=0}^{m} C_n \, z^n + \mathcal{O}(z^{m+1}),$$ (7)

where the coefficients C_n as a function of the c_n are uniquely determined (see, e.g., Eqs. (36) and (37) of [6]). We define partial sum of the Borel transform, expressed as a function of the conformal variable z, as

$$f_{\mathrm{B}}^{\prime(\lambda),m}(z) = \sum_{n=0}^{m} C_n \, z^n.$$ (8)

In a previous investigation [6], Caprini and Fischer evaluate the following transforms,

$$\mathcal{T}_m' f(g) = \frac{1}{g^\lambda} \int_0^\infty \mathrm{d}u \, u^{\lambda-1} \exp\big(-u/g\big) f_{\mathrm{B}}^{\prime(\lambda),m}(z(u)).$$ (9)

Caprini and Fischer [6] observe the apparent numerical convergence with increasing m. The limit as $m \to \infty$, provided it exists, is then assumed to represent the complete, physically relevant solution,

$$f(g) = \lim_{m \to \infty} \mathcal{T}_m' f(g).$$ (10)

We do not consider the question of the existence of this limit here (for an outline of questions related to these issues we refer to [7]).

3

In the absence of further information on the analyticity domain of the Borel transform (2), we cannot necessarily conclude that $f_{\rm B}^{(\lambda)}(u(z))$ as a function of z is analytic inside the unit circle of the complex z-plane, or that, for example, the conditions of Theorem 5.2.1 of [20] are fulfilled. Therefore, we propose a modification of the transforms (9). In particular, we advocate the evaluation of (lower-diagonal) Padé approximants [20, 21] to the function $f_{\rm B}^{\prime(\lambda),m}(z)$, expressed as a function of z,

$$f_{\rm B}^{\prime\prime(\lambda),m}(z) = \left[[\![m/2]\!] \Big/ [\![(m+1)/2]\!] \right]_{f_{\rm B}^{\prime(\lambda),m}}(z)\,. \tag{11}$$

We define the following transforms,

$$\mathcal{T}_m'' f(g) = \frac{1}{g^\lambda} \int_{C_j} {\rm d}u\, u^{\lambda-1}\, \exp\!\left(-u/g\right) f_{\rm B}^{\prime\prime(\lambda),m}\!\left(z(u)\right) \tag{12}$$

where the integration contour C_j ($j = -1, 0, 1$) have been defined in [8]. These integration contours have been shown to to provide the physically correct analytic continuation of resummed perturbation series for those cases where the evaluation of the standard Laplace-Borel integral (3) is impossible due to an insufficient analyticity domain of the integrand (possibly due to multiple branch cuts) or due to spurious singularities in view of the finite order of the Padé approximations defined in (11). We should mention potential complications due to multi-instanton contributions, as discussed for example in Ch. 43 of [2] (these are not encountered in the current investigation). In this letter, we use exclusively the contour C_0 which is defined as the half sum of the contours C_{-1} and C_{+1} displayed in Fig. 1 in [8]. At increasing m, the limit as $m \to \infty$, provided it exists, is then again assumed to represent the complete, physically relevant solution,

$$f(g) = \lim_{m \to \infty} \mathcal{T}_m'' f(g)\,. \tag{13}$$

Because we take advantage of the special integration contours C_j, analyticity of the Borel transform $f_{\rm B}^{(\lambda)}(u(z))$ inside the unit circle of the complex z-plane is not required, and additional acceleration of the convergence is mediated by employing Padé approximants in the conformal variable z.

Table 1: Resummation of the perturbation series (14) for the anomalous dimension γ function of the 6-dimensional ϕ^3 theory by the method defined in Eqs. (1)–(12). The transforms $\mathcal{T}_m'' \gamma_{\rm hopf}(g)$ are shown in the transformation order $m = 28,\ 29,\ 30$. The coupling g assumes the values $g = 5.0,\ 5.5,\ 6.0$ and $g = 10.0$.

m	$\mathcal{T}_m'' \gamma_{\rm hopf}(5.0)$	$\mathcal{T}_m'' \gamma_{\rm hopf}(5.5)$	$\mathcal{T}_m'' \gamma_{\rm hopf}(6.0)$	$\mathcal{T}_m'' \gamma_{\rm hopf}(10.0)$
28	$-0.501\ 565\ 232$	$-0.538\ 352\ 234$	$-0.573\ 969\ 740$	$-0.827\ 506\ 173$
29	$-0.501\ 565\ 232$	$-0.538\ 352\ 233$	$-0.573\ 969\ 738$	$-0.827\ 506\ 143$
30	$-0.501\ 565\ 231$	$-0.538\ 352\ 233$	$-0.573\ 969\ 738$	$-0.827\ 506\ 136$

We consider the resummation of two particular perturbation series discussed in [5] for the anomalous dimension γ function of the ϕ^3 theory in 6 dimensions and the Yukawa coupling in 4 dimensions. The perturbation series for the ϕ^3 theory is given in Eq. (16) in [5],

$$\gamma_{\rm hopf}(g) \sim \sum_{n=1}^{\infty} (-1)^n\, \frac{G_n}{6^{2n-1}}\, g^n\,, \tag{14}$$

Table 2: Resummation of the perturbation series (18) for the anomalous dimension of the Yukawa coupling via the method defined in Eqs. (1)–(12). The transforms $\mathcal{T}_m'' \tilde{\gamma}_{\mathrm{hopf}}(g)$ are shown in the order of transformation $m = 28,\ 29,\ 30$. For the coupling g, we consider the values $g = 5.0,\ 5.5,\ 6.0$ and $g = 30^2/(4\pi)^2 = 5.69932\ldots$.

m	$\mathcal{T}_m'' \tilde{\gamma}_{\mathrm{hopf}}(5.0)$	$\mathcal{T}_m'' \tilde{\gamma}_{\mathrm{hopf}}(5.5)$	$\mathcal{T}_m'' \tilde{\gamma}_{\mathrm{hopf}}(6.0)$	$\mathcal{T}_m'' \tilde{\gamma}_{\mathrm{hopf}}\left(30^2/(4\pi)^2\right)$
28	$-1.669\ 071\ 213$	$-1.800\ 550\ 588$	$-1.928\ 740\ 624$	$-1.852\ 027\ 809$
29	$-1.669\ 071\ 214$	$-1.800\ 550\ 589$	$-1.928\ 740\ 626$	$-1.852\ 027\ 810$
30	$-1.669\ 071\ 214$	$-1.800\ 550\ 589$	$-1.928\ 740\ 625$	$-1.852\ 027\ 810$

where the coefficients G_n are given in Table 1 in [5] for $n = 1, \ldots, 30$ (the G_n are real and positive). We denote the coupling parameter a used in [5] as g; this is done in order to ensure compatibility with the general power series given in Eq. (1). Empirically, Broadhurst and Kreimer derive the large-order asymptotics

$$G_n \sim \mathrm{const.} \times 12^{n-1}\, \Gamma(n+2)\,, \qquad n \to \infty\,, \tag{15}$$

by investigating the explicit numerical values of the coefficients $G_1, \ldots, G_{30}$. The leading asymptotics of the perturbative coefficients c_n are therefore (up to a constant prefactor)

$$c_n \sim (-1)^n \frac{\Gamma(n+2)}{3^n}\,, \qquad n \to \infty\,. \tag{16}$$

This implies that the λ-parameter in the Borel transform (2) should be set to $\lambda = 2$ (see also the notion of an asymptotically optimized Borel transform discussed in [4]). In view of Eq. (16), the pole closest to the origin of the Borel transform (2) is expected at

$$u = u_0^{\mathrm{hopf}} = -3\,, \tag{17}$$

and a rescaling of the Borel variable $u \to 3\,u$ in Eq. (3) then leads to an expression to which the method defined in Eqs. (1)–(12) can be applied directly. For the Yukawa coupling, the γ-function reads

$$\tilde{\gamma}_{\mathrm{hopf}}(g) \sim \sum_{n=1}^{\infty} (-1)^n \frac{\tilde{G}_n}{2^{2n-1}}\, g^n\,, \tag{18}$$

where the $\tilde{G}_n$ are given in Table 2 in [5] for $n = 1, \ldots, 30$. Empirically, i.e. from an investigation of the numerical values of $\tilde{G}_1, \ldots, \tilde{G}_{30}$, the following factorial growth in large order is derived [5],

$$\tilde{G}_n \sim \mathrm{const.}' \times 2^{n-1}\, \Gamma(n+1/2)\,, \qquad n \to \infty\,. \tag{19}$$

This leads to the following asymptotics for the perturbative coefficients (up to a constant prefactor),

$$c_n \sim (-1)^n \frac{\Gamma(n+1/2)}{2^n}\,, \qquad n \to \infty\,. \tag{20}$$

This implies that an asymptotically optimal choice [4] for the λ-parameter in (2) is $\lambda = 1/2$. The first pole of the Borel transform (2) is therefore expected at

$$u = \tilde{u}_0^{\mathrm{hopf}} = -2\,. \tag{21}$$

5

A rescaling of the Borel variable according to $u \to 2\,u$ in (3) enables the application of the resummation method defined in Eqs. (1)–(12).

In Table 1, numerical values for the transforms $\mathcal{T}''_m \gamma_{\text{hopf}}(g)$ are given, which have been evaluated according to Eq. (12). The transformation order is in the range $m = 28$, 29, 30, and we consider coupling parameters $g = 5.0$, 5.5, 6.0 and $g = 10.0$. The numerical values of the transforms display apparent convergence to about 9 significant figures for $g \leq 6.0$ and to about 7 figures for $g = 10.0$. In Table 2, numerical values for the transforms $\mathcal{T}''_m \tilde{\gamma}_{\text{hopf}}(g)$ calculated according to Eq. (12) are shown in the range $m = 28$, 29, 30 for (large) coupling strengths $g = 5.0$, 5.5, 6.0. Additionally, the value $g = 30^2/(4\,\pi)^2 = 5.69932\ldots$ is considered as a special case (as it has been done in [5]). Again, the numerical values of the transforms display apparent convergence to about 9 significant figures. At large coupling $g = 12.0$, the apparent convergence of the transforms suggests the following values: $\gamma_{\text{hopf}}(12.0) = -0.939\,114\,3(2)$ and $\tilde{\gamma}_{\text{hopf}}(12.0) = -3.287\,176\,9(2)$. The numerical results for the Yukawa case, i.e. for the function $\tilde{\gamma}_{\text{hopf}}$, have recently been confirmed by an improved analytic, nonperturbative investigation [22] which extends the perturbative calculation [5].

We note that the transforms $\mathcal{T}'_m \gamma_{\text{hopf}}(g)$ and $\mathcal{T}'_m \tilde{\gamma}_{\text{hopf}}(g)$ calculated according to Eq. (9), i.e. by the unmodified conformal mapping, typically exhibit apparent convergence to 5–6 significant figures in the transformation order $m = 28$, 29, 30 and at large coupling $g \geq 5$. Specifically, the numerical values for $g = 5.0$ are

$$\mathcal{T}'_{28} \gamma_{\text{hopf}}(g = 5.0) \quad = \quad -0.501\,567\,294\,,$$

$$\mathcal{T}'_{29} \gamma_{\text{hopf}}(g = 5.0) \quad = \quad -0.501\,564\,509\,,$$

$$\mathcal{T}'_{30} \gamma_{\text{hopf}}(g = 5.0) \quad = \quad -0.501\,563\,626\,.$$

These results, when compared to the data in Table 1, exemplify the acceleration of the convergence by the additional Padé approximation of the Borel transform *expressed as a function of the conformal variable* [see Eq. (11)].

It is not claimed here that the resummation method defined in Eqs. (1)–(12) necessarily provides the fastest possible rate of convergence for the perturbation series defined in Eq. (14) and (18). Further improvements should be feasible, especially if particular properties of the input series are known and exploited (see in part the methods described in [4]). We also note possible improvements based on a large-coupling expansion [23], in particular for excessively large values of the coupling parameter g, or methods based on order-dependent mappings (see [11, 12] or the discussion following Eq. (41.67) in [2]).

The conformal mapping [6, 7] is capable of accomplishing the analytic continuation of the Borel transform (2) beyond the circle of convergence. Padé approximants, applied directly to the partial sums of the Borel transform (4), provide an alternative to this method [4, 5, 8–10]. Improved rates of convergence can be achieved when the convergence of the transforms obtained by conformal mapping in Eq. (7) is accelerated by evaluating Padé approximants as in Eq. (11), and conditions on analyticity domains can be relaxed in a favorable way when these methods are combined with the integration contours from Ref. [8]. Numerical results for the resummed values of the perturbation series (14) and (18) are provided in the Tables 1 and 2. By the improved conformal mapping and other optimized resummation techniques (see, e.g., the methods introduced in Ref. [4]) the applicability of perturbative (small-coupling) expansions can be generalized to the regime of large coupling and still lead to results of relatively high accuracy.

References

[1] J. C. LeGuillou and J. Zinn-Justin, *Large-Order Behaviour of Perturbation Theory* (North-Holland, Amsterdam, 1990).

[2] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, 3rd ed. (Clarendon Press, Oxford, 1996).

[3] J. Fischer, Int. J. Mod. Phys. A **12**, 3625 (1997).

[4] U. D. Jentschura, E. Weniger, and G. Soff, Asymptotic Improvement of Resummation and Perturbative Predictions, Los Alamos preprint hep-ph/0005198, submitted.

[5] D. Broadhurst and D. Kreimer, Phys. Lett. B **475**, 63 (2000).

[6] I. Caprini and J. Fischer, Phys. Rev. D **60**, 054014 (1999).

[7] I. Caprini and J. Fischer, Convergence of the expansion of the Laplace-Borel integral in perturbative QCD improved by conformal mapping, Los Alamos preprint hep-ph/0002016.

[8] U. D. Jentschura, Resummation of Nonalternating Divergent Perturbative Expansions, Los Alamos preprint hep-ph/0001135, Phys. Rev. D (in press).

[9] P. A. Rączka, Phys. Rev. D **43**, R9 (1991).

[10] M. Pindor, Padé Approximants and Borel Summation for QCD Perturbation Series, Los Alamos preprint hep-th/9903151.

[11] R. Seznec and J. Zinn-Justin, J. Math. Phys. **20**, 1398 (1979).

[12] J. C. L. Guillou and J. Zinn-Justin, Ann. Phys. (N. Y.) **147**, 57 (1983).

[13] R. Guida, K. Konishi, and H. Suzuki, Ann. Phys. (N. Y.) **241**, 152 (1995).

[14] D. J. Broadhurst, P. A. Baikov, V. A. Ilyin, J. Fleischer, O. V. Tarasov, and V. A. Smirnov, Phys. Lett. B **329**, 103 (1994).

[15] G. Altarelli, P. Nason, and G. Ridolfi, Z. Phys. C **68**, 257 (1995).

[16] D. E. Soper and L. R. Surguladze, Phys. Rev. D **54**, 4566 (1996).

[17] K. G. Chetyrkin, J. H. Kühn, and M. Steinhauser, Phys. Lett. B **371**, 93 (1996).

[18] K. G. Chetyrkin, J. H. Kühn, and M. Steinhauser, Nucl. Phys. B **482**, 213 (1996).

[19] K. G. Chetyrkin, R. Harlander, and M. Steinhauser, Phys. Rev. D **58**, 014012 (1998).

[20] G. A. Baker and P. Graves-Morris, *Padé approximants*, 2nd ed. (Cambridge University Press, Cambridge, 1996).

[21] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill, New York, NY, 1978).

[22] D. Broadhurst and D. Kreimer, in preparation (2000).

[23] E. J. Weniger, Phys. Rev. Lett. **77**, 2859 (1996).